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Integrable hamiltonian for classical strings on $\mathrm{AdS}_5 \times \mathrm{S}^5$

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Abstract: We find the hamiltonian for physical excitations of the classical bosonic string propagating in the $\mathrm{AdS}_5 \times \mathrm{S}^5$ space-time. The hamiltonian is obtained in a so-called uniform gauge which is related to the static gauge by a 2d duality transformation. The hamiltonian is of the Nambu type and depends on two parameters: a single S^5 angular momentum J and the string tension λ. In the general case both parameters can be finite. The space of string states consists of short and long strings. In the sector of short strings the large J expansion with $\lambda' = \lambda/J^2$ fixed recovers the plane-wave hamiltonian and higher-order corrections recently studied in the literature. In the strong coupling limit $\lambda \to \infty$, J fixed, the energy of short strings scales as $\sqrt[4]{\lambda}$ while the energy of long strings scales as $\sqrt{\lambda}$. We further show that the gauge-fixed hamiltonian is integrable by constructing the corresponding Lax representation. We discuss some general properties of the monodromy matrix, and verify that the asymptotic behavior of the quasi-momentum perfectly agrees with the one obtained earlier for some specific cases.

Keywords: Sigma Models, Penrose limit and pp-wave background, AdS-CFT and dS-CFT Correspondence.

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Contents

1. Introduction

Our understanding of the gauge/string duality [1] has recently improved due to new ideas and techniques on both sides of the AdS/CFT correspondence. Even though the duality is of strong/weak coupling type, it was proposed by Berenstein, Maldacena and Nastase (BMN) [2] that energies of certain string states can be matched with perturbative scaling dimensions of dual SYM operators. The BMN results were re-interpreted in [3] as the semi-classical quantization near a point-like string carrying large momentum J along the central circle of S^5. The BMN proposal was then generalized in [4] where it was found by using the semi-classical approach that there exists a large sector of highly energetic string states on $\mathrm{AdS}_5 \times \mathrm{S}^5$ which permits a direct comparison with perturbative gauge theory. Any quantum "heavy" state can be well approximated by a classical string. However, in most cases the string energy turns out to be non-analytic in the 't Hooft coupling constant λ [3], that precludes a direct comparison with perturbative gauge theory.

Quite remarkably, as was shown in [4], energies of classical multi-spin strings rapidly rotating in S^5 admit an expansion in integer powers of the effective coupling constant λ/L^2, where L is the large, total spin on S^5. Even though the coupling λ is large in the semi-classical approach, this expansion is of the same form as in perturbative gauge theory, and, therefore, one may compare the energies of spinning strings with perturbative scaling dimensions of gauge theory operators.

The multi-spin string solutions are completely determined by the bosonic part of the Green-Schwarz superstring action [5]. In the conformal gauge the bosonic string is described by the sigma model with the $\mathrm{AdS}_5 \times \mathrm{S}^5$ target space, which is known to be exactly solvable. A finite-dimensional reduction of the classical string sigma model to an integrable system of the Neumann type [6] describes folded and circular rigid strings. More general string solutions are described by integral equations of the Bethe type. For strings moving in $\mathbb{R} \times \mathrm{S}^3$, $\mathrm{AdS}_3 \times \mathrm{S}^1$ and $\mathbb{R} \times \mathrm{S}^5$ they were derived in [7].

A related development in gauge theory was triggered by an important observation [8] that in the SO(6) subsector planar superconformal $\mathcal{N} = 4$ SYM is an integrable system in the one-loop approximation. This was generalized to the complete dilatation operator in [9] and, the integrability, very likely, holds at higher loops as well [10]. Integrable structures of QCD were previously observed in [11]. Integrability allows one to formulate a system of Bethe equations which is then used to find anomalous dimensions of conformal operators. For a closed $\mathfrak{su}(2)$ subsector the one-loop Bethe ansatz of [8] is extended up to three loops [12] owing to the fact that the corresponding three-loop dilatation operator [10] can be embedded into the Inozemtsev long-range spin chain [13]. Recently the all-loop asymptotic Bethe ansatz for the dilatation operator acting in the $\mathfrak{su}(2)$ subsector was proposed in [14].

The spin chain Bethe equations were used in [15, 12] to demonstrate one- and two-loop agreement between gauge and string theory predictions in the cases of folded and circular rigid strings. Moreover, as was shown in [16], the eigenvalues of higher local commuting charges also agree up to two-loop order, indicating a close relation between integrable structures of gauge and string theories. Furthermore, up to the second order of perturbation theory, the string Bethe equations [7] coincide with the spin chain Bethe equations [12] thus leading to a proof of two-loop agreement of string and gauge theory results in the $\mathfrak{su}(2)$ subsector. The one- and two-loop agreement in various subsectors of the gauge theory was also demonstrated in [17]–[21]. The other relevant aspects of the gauge/string duality have been investigated in [22, 23].

At two leading orders of perturbation theory the matching between gauge and string theory quantities was also observed for $1/L$ corrections to the BMN limit [24, 25]. It was noticed, however, that it breaks down at three-loop order [25]. The same pattern, i.e. the one- and two-loop agreement and disagreement starting at three loops, was also found for spinning strings [12, 16]. Moreover, results of [26, 27] indicate that $1/L$ corrections for spinning strings would disagree already at the one-loop level. As was stressed in [12, 14], the origin of all these disagreements could be due to neglecting on the gauge theory side the so-called wrapping interactions, and the agreement might be restored after they would have been properly incorporated. Let us also note that disagreement between gauge and string theories becomes manifest if one compares the thermodynamic limit of the all-loop asymptotic Bethe ansatz [14] describing (infinitely) long operators, and the string Bethe equations [7] describing classical spinning strings.

Assuming the validity of the AdS/CFT correspondence one should expect existence of a Bethe ansatz for quantum strings which would serve as a discretization of the integral (continuous) Bethe equations for classical strings and, from the gauge theory perspective, include terms responsible for wrapping interactions. An interesting discretization of the string Bethe equations was recently proposed in [28]. The Bethe ansatz for quantum

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strings [28] reproduces the near BMN spectrum of [25], the famous $\sqrt[4]{\lambda}$ behavior at strong coupling [29], and has a spin chain description at weak coupling [30]. The general multi-impurity spectrum (in the $\mathfrak{su}(2)$ subsector) predicted in [28] has been recently reproduced from the quantized string theory in the near plane-wave background [31]. Existence of a Bethe ansatz with such remarkable properties provides a strong evidence in favor of integrability of quantum strings. An interesting recent discussion of the quantum integrability for strings in the near plane-wave background ($1/J$ order) can be found in [32].

A necessary (but not always sufficient) condition for classical integrability of a solvable model is the existence of a Lax pair. The Lax representation implies the existence of an infinite number of local conserved charges that may allow one to solve the system exactly. A Lax pair for the classical superstring theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$ was found in [33] (see also [34]). To analyze quantum integrability of the superstring theory one would need to develop the hamiltonian formalism. The Poisson bracket of the $\mathscr{L}$-operator entering the Lax pair determines a classical r-matrix which is further used to find the Poisson algebra of the corresponding monodromy operator, and to quantize the model [35]. The knowledge of the hamiltonian structure is also crucial to exhibit commutativity of local integrals of motion. Unfortunately, the Lax pair of [33] cannot be immediately used to find an r-matrix structure and, therefore, to quantize the model because the local κ-symmetry has not been fixed, and, as the consequence, the Poisson structure of the $\mathscr{L}$-operator remains to be undefined. Also, to find the Lax pair, the two-dimensional metric on the string world-sheet has been fixed in a way equivalent to fixing the conformal gauge. It is well-known that in the conformal gauge fixing κ-symmetry leads to a very complicated Poisson structure for fermions which can be hardly used to quantize superstring even in flat space. Of course, the standard way to overcome this difficulty is to further impose the light-cone gauge. For string theory on a curved background, however, the usual conformal gauge and the light-cone gauge are not necessary compatible, fixing the light-cone gauge leads to a modification of the conformal gauge condition [36, 37].

Following the analogy with the Green-Schwarz superstring in flat space, it seems reasonable to use a light-cone type gauge to address the problems of integrability and quantization of superstring theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$. As a first step in this direction, in the present paper we consider the bosonic part of the superstring theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$ in such a gauge.

As is known, the direct quantization of superstrings in the *near* plane-wave background can be performed [25, 24] by using exact solvability of the superstring theory on plane waves [38]. In particular, the approach undertaken in [25] yields (perturbatively) the string hamiltonian as a power series in $1/J$. It appears, however, that there is another choice of a gauge condition which enables one to determine the bosonic part of the string hamiltonian as an exact function of J. This gauge condition is similar to the uniform gauge used in [19] which is related to the static gauge by a 2d duality transformation [20]. The uniform gauge uses the gauge freedom to request that the target space-time would coincide with the world-sheet time, and that one combination of the global R-symmetry charges would be homogeneously distributed along the string. The only difference of our gauge choice from the one used in [19] is that the authors of [19] distribute the total S^5 angular momentum $J_1 + J_2 + J_3$, while we only distribute a single component $J = J_3$ of the S^5

angular momentum. Therefore, in our gauge we study a sector of string states with one angular momentum fixed, and in the large J limit we should expect to recover the light-cone plane-wave hamiltonian. The uniform gauge we use is in fact a proper hamiltonian version of the light-cone gauge of [25]. In our consideration we keep two parameters J and λ finite. By this reason we can study not only short strings which in the BMN limit, $J \to \infty$, λ/J^2 fixed, represent small fluctuations around the point-like string carrying large momentum J along the central circle of S^5, but also strings which wind around a circle of S^5 and remain long even in the BMN limit. Rigid long string configuration were studied in [6]. In the sector of short strings the expansion in $1/J$ of our hamiltonian reproduces the plane-wave hamiltonian and the $1/J$ and $1/J^2$ corrections obtained in [25, 39]. We believe that it should be possible to take into account the fermionic degrees of freedom and obtain a finite J, λ hamiltonian in the uniform gauge for Green-Schwarz superstring on $\mathrm{AdS}_5 \times \mathrm{S}^5$.

Since both parameters J and λ are finite one could try to consider the strong coupling limit $\lambda \to \infty$, J fixed. It turns out that the hamiltonian does not have any good large λ expansion neither in the sector of short strings nor in the sector of long strings. Nevertheless, one can see the famous $\sqrt[4]{\lambda}$ leading behavior of the string energy in the sector of short strings. The long strings, however, are much heavier in the strong coupling limit, and their energy scales as $\sqrt{\lambda}$.

In our gauge the hamiltonian is of the Nambu type,[1] and it may seem difficult to obtain a Lax representation for it. On the other hand, in the conformal gauge the string model is described as a reduction of the well-known principle sigma model to the coset space $\mathrm{AdS}_5 \times \mathrm{S}^5$. In the uniform gauge the world-sheet metric is non-diagonal and depends non-trivially on the physical fields. Nevertheless, the Lax representation for the principle model on arbitrary 2d surface can be easily constructed and further used to derive the Lax pair for the uniform gauge hamiltonian. This proves the (kinematical) integrability of the hamiltonian. We would like to emphasize that our method is universal and can be applied to derive a Lax representation for any gauge-fixed hamiltonian, in particular, for the AdS_5 light-cone hamiltonian obtained in [37].

The paper is organized as follows. In section 2 we derive the physical hamiltonian in the uniform gauge, and express the world-sheet metric in terms of physical degrees of freedom. In section 3 we discuss the (near) BMN and strong coupling limits. In section 4 we obtain the Lax representation for the physical hamiltonian. In section 5 we discuss some general properties of the monodromy matrix. We show, in particular, how some of the results obtained in [7] can be re-derived and generalized within our approach. In appendices we collect some useful formulae. In particular, in appendix B we specify our general treatment to the case of classical strings moving in $\mathbb{R} \times \mathrm{S}^3$.

2. Gauge-fixed hamiltonian

In this section we develop the hamiltonian formalism for strings in the uniform gauge. We start with describing suitable parametrizations of the sphere and the AdS spaces. The

[1] A similar but different light-cone hamiltonian of the square root type was also obtained for some specific choice of the light-cone coordinates in the second paper of [25].

five-sphere can be parametrized by five variables: $y_i, i = 1, \dots, 4$ and by the angle variable ϕ. In terms of six real embedding coordinates Y_A, $A = 1, \dots, 6$ obeying the condition $Y_A^2 = 1$ the parametrization reads

$$\mathcal{Y}_1 \equiv Y_1 + iY_2 = \frac{y_1 + iy_2}{1 + \frac{y^2}{4}}, \qquad \mathcal{Y}_2 \equiv Y_3 + iY_4 = \frac{y_3 + iy_4}{1 + \frac{y^2}{4}},$$
$$\mathcal{Y}_3 \equiv Y_5 + iY_6 = \frac{1 - \frac{y^2}{4}}{1 + \frac{y^2}{4}} \exp(i\phi).$$

The metric induced on S^5 from the flat metric of the embedding space is

$$dY_A dY_A = \left(\frac{1 - \frac{y^2}{4}}{1 + \frac{y^2}{4}} \right)^2 d\phi^2 + \frac{dy_i dy_i}{(1 + \frac{y^2}{4})^2}.$$

Here and below we use the concise notation $y^2 = y_i y_i$. Analogously to describe the five-dimensional AdS space we introduce four coordinates z_i and the global AdS time t. The embedding coordinates Z_A, which obey $\eta_{AB} Z^A Z^B = -1$ with the metric $\eta_{AB} = (-1, 1, 1, 1, 1, -1)$, are now parametrized as

$$\mathcal{Z}_1 \equiv Z_1 + iZ_2 = -\frac{z_1 + iz_2}{1 - \frac{z^2}{4}}, \qquad \mathcal{Z}_2 \equiv Z_3 + iZ_4 = -\frac{z_3 + iz_4}{1 - \frac{z^2}{4}},$$
$$\mathcal{Z}_3 \equiv Z_0 + iZ_5 = \frac{1 + \frac{z^2}{4}}{1 - \frac{z^2}{4}} \exp(it).$$

For the induced metric one obtains

$$\eta_{AB} dZ^A dZ^B = -\left(\frac{1 + \frac{z^2}{4}}{1 - \frac{z^2}{4}} \right)^2 dt^2 + \frac{dz_i dz_i}{(1 - \frac{z^2}{4})^2}.$$

The same parametrization of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ space was also discussed in the context of $\mathrm{AdS}_5 \times \mathrm{S}^5$ string quantization in the first paper of [4] and in [25].

Since we consider closed strings all the fields Y_A and Z_A are assumed to be periodic functions of the world-sheet coordinate $0 \le \sigma \le 2\pi$. Periodicity implies that the angle variable ϕ has to satisfy the constraint:

$$\phi(2\pi) - \phi(0) = -2\pi m, \qquad m \in \mathbb{Z}. \tag{2.1}$$

The integer number m represents the number of times the string winds around the circle parametrized by ϕ.

Propagation of bosonic string is described by the sigma model with $\mathrm{AdS}_5 \times \mathrm{S}^5$ target space. The corresponding lagrangian density reads

$$\mathcal{L} = -\frac{1}{2}\sqrt{\lambda}\, \gamma^{\alpha\beta} \left(-G_{tt} \partial_\alpha t \partial_\beta t + \frac{\partial_\alpha z_i \partial_\beta z_i}{(1 - \frac{z^2}{4})^2} + G_{\phi\phi} \partial_\alpha \phi \partial_\beta \phi + \frac{\partial_\alpha y_i \partial_\beta y_i}{(1 + \frac{y^2}{4})^2} \right). \tag{2.2}$$

Here $\gamma^{\alpha\beta} \equiv \sqrt{-h} h^{\alpha\beta}$, where $h_{\alpha\beta}$ is a world-sheet metric with Minkowski signature. We also introduced two functions G_{tt} and $G_{\phi\phi}$:

$$G_{tt} = \left(\frac{1+\frac{z^2}{4}}{1-\frac{z^2}{4}} \right)^2 , \qquad G_{\phi\phi} = \left(\frac{1-\frac{y^2}{4}}{1+\frac{y^2}{4}} \right)^2 . \tag{2.3}$$

The string tension $\sqrt{\lambda}$ is related to the radius R of S^5 (AdS_5) and the slope α' of the Regge trajectory as $\sqrt{\lambda} = \frac{R^2}{\alpha'}$.

A consistent quantization procedure would require finding the true dynamical (physical) variables for the string sigma model. The most elegant way to achieve this goal is to use the hamiltonian formulation. Deriving from eq. (2.2) the canonical momenta for all the fields we recast the lagrangian in the phase space form

$$\begin{aligned} \mathcal{L} = {} & p_t \dot{t} + p_\phi \dot{\phi} + p_{z_i} \dot{z}_i + p_{y_i} \dot{y}_i - \\ & - \frac{1}{2\sqrt{\lambda}\, \gamma^{\tau\tau}} \left(\frac{p_t^2}{G_{tt}} - \frac{p_\phi^2}{G_{\phi\phi}} + \lambda G_{tt} t'^2 - \lambda G_{\phi\phi} \phi'^2 - \right. \\ & \left. \qquad - \left(1-\frac{z^2}{4}\right)^2 p_{z_i}^2 - \left(1+\frac{y^2}{4}\right)^2 p_{y_i}^2 - \frac{\lambda z_i'^2}{(1-\frac{z^2}{4})^2} - \frac{\lambda y_i'^2}{(1+\frac{y^2}{4})^2} \right) + \\ & + \frac{\gamma^{\tau\sigma}}{\gamma^{\tau\tau}} \left(p_t t' + p_\phi \phi' + p_{z_i} z_i' + p_{y_i} y_i' \right) . \end{aligned} \tag{2.4}$$

Here p_t, p_ϕ, p_{z_i} and p_{y_i} are the canonical momenta conjugate to t, ϕ, z_i and y_i respectively. The dot means the derivative with respect to the world-sheet time τ while prime denotes the derivative with respect to σ. Note also that to derive the formula we used that the definition of $\gamma^{\alpha\beta}$ implies $\det \gamma^{-1} = -1$. In what follows we will often use the shorthand notation for pairings: $p_{z_i} z_i' = p_z z'$, etc.

The uniform gauge we want to fix is of the type considered in [19, 20], and consists in imposing the following two conditions[2]

$$t = \tau , \qquad p_\phi = J . \tag{2.5}$$

Equations of motion for the phase space variables are found from eq. (2.4). Upon further substitution of the gauge conditions (2.5) some of these equations turn into the constraints. Solving the constraints allows one to exclude the gauge degrees of freedom, and obtain the hamiltonian for physical variables. Let us now discuss this procedure in more detail.

First by varying the lagrangian (2.4) with respect to $\gamma^{\tau\sigma}$ we find an equation to determine ϕ' (note that $t' = 0$):

$$\phi' = -\frac{1}{J}(p_z z' + p_y y') . \tag{2.6}$$

Integrating this equation over σ and recalling eq. (2.1) we obtain a constraint

$$\mathcal{V} = m\, J , \tag{2.7}$$

[2]The same gauge can be also used to analyze the superstring theory on $AdS_5 \times S^5$. The main technical complication, then, is in fixing the κ-symmetry and finding the canonical fermionic variables.

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where

$$\mathcal{V} = \int_0^{2\pi} \frac{\mathrm{d}\sigma}{2\pi}(p_z z' + p_y y')\,. \tag{2.8}$$

In the string language this residual constraint is nothing else but the level-matching condition. We will not try to solve eq. (2.7) in classical theory, rather, following the analogy with the flat case, we will require eq. (2.7) to be satisfied by physical states of the theory.

The variable p_t conjugate to the global AdS time t is the density of the space-time energy of string. On the other hand, fixing $t = \tau$ allows one to identify $-p_t$ with the hamiltonian density $\mathcal{H}$ for physical degrees of freedom:

$$\mathrm{H} = \int_0^{2\pi} \frac{\mathrm{d}\sigma}{2\pi}\mathcal{H}\,. \tag{2.9}$$

To obtain $p_t^2 \equiv \mathcal{H}^2$ one performs variation of eq. (2.4) with respect to $\gamma^{\tau\tau}$ and use eq. (2.6). In this way we find the square of the hamiltonian density $\mathcal{H}^2$

$$\begin{aligned}\mathcal{H}^2 \;=\; & \frac{G_{tt}}{G_{\phi\phi}}J^2 + \frac{\lambda}{J^2}G_{tt}G_{\phi\phi}(p_z z' + p_y y')^2 + G_{tt}\left(1-\frac{z^2}{4}\right)^2 p_z^2 + \lambda\frac{G_{tt}}{\left(1-\frac{z^2}{4}\right)^2}z'^2 + \\ & + G_{tt}\left(1+\frac{y^2}{4}\right)^2 p_y^2 + \lambda\frac{G_{tt}}{\left(1+\frac{y^2}{4}\right)^2}y'^2\,. \end{aligned} \tag{2.10}$$

We therefore see that the physical variables of our theory are the eight coordinates y, z and the corresponding conjugate momenta p_y, and p_z. They are subject of the canonical Poisson brackets

$$\begin{aligned}\{p_{z_i}(\sigma), z_j(\sigma')\} &= 2\pi\delta_{ij}\delta(\sigma-\sigma') \\ \{p_{y_i}(\sigma), y_j(\sigma')\} &= 2\pi\delta_{ij}\delta(\sigma-\sigma')\,. \end{aligned} \tag{2.11}$$

It is clear that the gauge-fixed theory is manifestly invariant under the SO(4) $\times$ SO(4) subgroup of the R-symmetry group of the string theory. The hamiltonian density is given by the square root and, therefore, the physical hamiltonian appears to be of the *Nambu type*. It depends on two parameters, J and λ. Thus, we have completely described the hamiltonian structure of the theory, the equation of motion for any function Φ of physical variables is given by

$$\dot{\Phi} = \{\mathrm{H}, \Phi\}\,. \tag{2.12}$$

Let us note that we have not yet exploit all the information contained in eq. (2.4). In particular, equation for p_t allows one to solve for $\gamma^{\tau\tau}$. Indeed,

$$0 = \frac{\delta\mathcal{L}}{\delta p_t} = 1 - \frac{1}{\gamma^{\tau\tau}}\frac{p_t}{\sqrt{\lambda}G_{tt}}\,, \tag{2.13}$$

i.e.

$$\gamma^{\tau\tau} = \frac{p_t}{\sqrt{\lambda}G_{tt}} = -\frac{\mathcal{H}}{\sqrt{\lambda}G_{tt}}\,. \tag{2.14}$$

Note again that to express the r.h.s. of the last formula via the hamiltonian density we picked up the negative root of the equation $p_t^2 = \mathcal{H}^2$. Such a prescription is dictated by positivity of the physical hamiltonian and by agreement with the plane-wave limit as will be discussed below. Finally, we can exploit an equation for ϕ

$$\frac{d}{d\tau}\frac{\delta\mathcal{L}}{\delta\dot{\phi}} - \frac{\delta\mathcal{L}}{\delta\phi} = -\partial_\sigma\left(\frac{\sqrt{\lambda}G_{\phi\phi}}{\gamma^{\tau\tau}}\phi' + J\frac{\gamma^{\tau\sigma}}{\gamma^{\tau\tau}}\right) = 0 \tag{2.15}$$

to solve for the metric component $\gamma^{\tau\sigma}$. One finds

$$\gamma^{\tau\sigma} = f(\tau)\frac{\mathcal{H}}{\sqrt{\lambda}\,G_{tt}} + \frac{\sqrt{\lambda}}{J^2}G_{\phi\phi}(p_z z' + p_y y')\,, \tag{2.16}$$

where $f(\tau)$ is an arbitrary function of τ. The presence of this function signals a residual symmetry. Indeed, in the lagrangian (2.7) we can shift the ratio $\frac{\gamma^{\tau\sigma}}{\gamma^{\tau\tau}}$ by any function $f(\tau)$. On the solutions of eq. (2.7) the lagrangian remains invariant under this shift. The function $f(\tau)$ plays the role of the lagrangian multiplier to the level-matching constraint $\mathcal{V}$. Thus, keeping a non-trivial $f(\tau)$ requires the following modification of the hamiltonian

$$\mathrm{H} \to \mathrm{H} + f(\tau)(\mathcal{V} - mJ)\,. \tag{2.17}$$

Of course, on solutions of the level-matching constraint this hamiltonian coincides with the old one. In what follows we pick up $f(\tau) = 0$, *i.e.*

$$\gamma^{\tau\sigma} = \frac{\sqrt{\lambda}}{J^2}G_{\phi\phi}(p_z z' + p_y y')\,. \tag{2.18}$$

Thus, the world-sheet metric is completely determined in terms of physical fields which is equivalent to solving the Virasoro constraints. We see that fixing the uniform gauge invokes a non-trivial gravitational field on the physical world-sheet. One can easily check that the only constraint $\mathcal{V}$, which we left unsolved, commutes with the hamiltonian:

$$\{\mathrm{H}, \mathcal{V}\} = 0 \tag{2.19}$$

and is, in fact, a generator of residual symmetry that generates rigid rotations: $\sigma \to \epsilon\sigma$:

$$\{\mathcal{V}, y(\sigma)\} = y'(\sigma)\,, \qquad \{\mathcal{V}, p(\sigma)\} = p'(\sigma)\,. \tag{2.20}$$

These equations are of the evolution type, *cf.* (2.12), where now σ plays the role of the (compact) time variable and the "hamiltonian" is $\mathcal{V}$. Thus, H and $\mathcal{V}$ generate two commuting hamiltonian flows of the dynamical variables corresponding to the times τ and σ.

Since the hamiltonian commutes with the level-matching constraint $\mathcal{V}$, the physical string states are divided into sectors labeled by the winding number m because the eigenvalues of $\mathcal{V}$ are equal to mJ, in accord with (2.7). In what follows we will loosely refer to strings with zero winding number as short strings, and to strings with $m \neq 0$ as long strings. Let us note, however, that since we consider strings moving in S^5, a string with a nonzero winding number in the ϕ-direction may still be of a small size if it is located near the coordinate singularity of S^5 ($y^2 \sim 4$).

3. Near BMN and strong coupling limits

In this section we use the gauge-fixed hamiltonian to discuss the BMN limit, $J \to \infty$ with λ/J^2 fixed, and the strong coupling limit, $\lambda \to \infty$ with J fixed. In the uniform gauge we study a sector of string states with one S^5 angular momentum fixed, therefore, in the BMN limit we should expect to recover the light-cone plane-wave hamiltonian, and the $1/J$ and $1/J^2$ corrections obtained in [25, 39].

The authors of [25] set up a procedure to construct a perturbative large-curvature expansion of the string hamiltonian on $\mathrm{AdS}_5 \times \mathrm{S}^5$ obtained in the light-cone gauge around the pp-wave background. In particular, they explicitly obtained the quartic and (even higher [39]) correction to the pp-wave hamiltonian and studied the problem of its diagonalization in quantum theory. The uniform gauge we chose is, in fact, a proper hamiltonian version of the light-cone gauge used in [2, 25] valid for finite J, λ. To make a connection to the pp-wave limit we rescale the coordinates as

$$z \to \frac{1}{\sqrt{J}} z\,, \qquad y \to \frac{1}{\sqrt{J}} y\,, \qquad p_z \to \sqrt{J} p_z\,, \qquad p_y \to \sqrt{J} p_y\,. \tag{3.1}$$

This rescaling is a canonical transformation because it preserves the canonical Poisson brackets (2.11). To write down the rescaled hamiltonian density we introduce an effective BMN coupling λ'

$$\lambda' = \frac{\lambda}{J^2}\,.$$

To take into account the level-matching condition (2.7) we write $(p_z z' + p_y y')^2$ in the hamiltonian (2.10) as $\mathcal{V}^2 + (p_z z' + p_y y')^2_*$, where $\mathcal{V} = mJ$ is the zero mode, and the second term with the subscript $*$ represents the terms depending on non-zero Fourier modes of $p_z z' + p_y y'$. Then the square of the density becomes

$$\begin{aligned}\mathcal{H}^2 = & \frac{G_{tt}}{G_{\phi\phi}} J^2 + \lambda' m^2 J^2 G_{tt} G_{\phi\phi} + \lambda' G_{tt} G_{\phi\phi} (p_z z' + p_y y')^2_* + \\ & + J G_{tt} \left(1 - \frac{z^2}{4J}\right)^2 p_z^2 + J\lambda' \frac{G_{tt}}{\left(1 - \frac{z^2}{4J}\right)^2} z'^2 + \\ & + J G_{tt} \left(1 + \frac{y^2}{4J}\right)^2 p_y^2 + J\lambda' \frac{G_{tt}}{\left(1 + \frac{y^2}{4J}\right)^2} y'^2\,,\end{aligned} \tag{3.2}$$

where the rescaled functions G_{tt} and $G_{\phi\phi}$ are given by

$$G_{tt} = \left(\frac{1 + \frac{z^2}{4J}}{1 - \frac{z^2}{4J}}\right)^2\,, \qquad G_{\phi\phi} = \left(\frac{1 - \frac{y^2}{4J}}{1 + \frac{y^2}{4J}}\right)^2\,. \tag{3.3}$$

Now we see that there are two principally different cases: (i) the case of short strings with the winding number around the circle parametrized by ϕ equal to zero, $m = 0$, and (ii) the case of long strings winding around the circle with $m \neq 0$. Then the large-curvature expansion around pp-wave is obtained in the sector of short strings by sending $J \to \infty$, while keeping the BMN coupling λ' finite. The leading terms of the large J expansion are

$$\mathcal{H} = J + \mathcal{H}_{\rm pp} + \cdots \,, \tag{3.4}$$

where the second term is a density for the pp-wave hamiltonian

$$\mathcal{H}_{\rm pp} = \frac{1}{2}(p_y^2 + p_z^2 + y^2 + z^2 + \lambda' y'^2 + \lambda' z'^2)\,. \tag{3.5}$$

Expanding further one can easily check that the terms of order $1/J$ and $1/J^2$ precisely agree with those found in [25, 39]. Thus, the advantage of our approach is that it allows us to obtain the physical hamiltonian for finite J and, therefore, to study its general properties, without appealing to perturbation theory. Also a perturbative expansion becomes easy to handle as it is now encoded in the unique expression (3.2).

According to (3.4) in the pp-wave limit $\mathcal{H} \to J$ and, therefore,

$$\gamma^{\tau\tau} \to -\frac{1}{\sqrt{\lambda'}}\,, \qquad \gamma^{\tau\sigma} \to 0 \tag{3.6}$$

which is essentially the flat metric, as it should be. This also motivates our choice of the sign in eq. (2.14).

In the sector of short strings the eigenvalues of the quadratic hamiltonian $\mathcal{H}_{\rm pp}$ acting on physical states satisfying the level-matching constraint (2.7) are of order 1. This is the reason why the $1/J$ perturbative expansion can be used in the sector of short strings. On the other hand in the sector of long strings with the winding number $m \neq 0$ the large J expansion cannot be used because in that case the corresponding quadratic hamiltonian has large eigenvalues of order J on the physical states, that makes the formal $1/J$ expansion meaningless. A proper large J expansion in the sector of long strings requires first to find a classical solution[3] satisfying the level-matching condition (2.7), and then expand around this solution following the lines discussed in [4].

It is also of interest to consider the strong coupling limit, $\lambda \to \infty$ with J fixed. In this case we rescale the coordinates as

$$z \to \frac{1}{\sqrt[4]{\lambda}} z\,, \qquad y \to \frac{1}{\sqrt[4]{\lambda}} y\,, \qquad p_z \to \sqrt[4]{\lambda} p_z\,, \qquad p_y \to \sqrt[4]{\lambda} p_y\,. \tag{3.7}$$

This rescaling is clearly a canonical transformation.[4] Then the square of the density takes the form

$$\begin{aligned}\mathcal{H}^2 \,=\, & \frac{G_{tt}}{G_{\phi\phi}} J^2 + \frac{\lambda}{J^2} G_{tt} G_{\phi\phi} (p_z z' + p_y y')^2 + \\ & + \sqrt{\lambda} G_{tt} \left(1 - \frac{z^2}{4\sqrt{\lambda}}\right)^2 p_z^2 + \sqrt{\lambda} \frac{G_{tt}}{\left(1 - \frac{z^2}{4\sqrt{\lambda}}\right)^2} z'^2 +\end{aligned}$$

[3] A large class of long string configurations was found in the second paper of [6].

[4] This rescaling also induces the rescaling (3.1) if λ' is finite.

$$+\sqrt{\lambda}G_{tt}\left(1+\frac{y^2}{4\sqrt{\lambda}}\right)^2 p_y^2+\sqrt{\lambda}\frac{G_{tt}}{\left(1+\frac{y^2}{4\sqrt{\lambda}}\right)^2}y'^2\,, \tag{3.8}$$

where the rescaled functions G_{tt} and $G_{\phi\phi}$ are given by

$$G_{tt}=\left(\frac{1+\frac{z^2}{4\sqrt{\lambda}}}{1-\frac{z^2}{4\sqrt{\lambda}}}\right)^2, \qquad G_{\phi\phi}=\left(\frac{1-\frac{y^2}{4\sqrt{\lambda}}}{1+\frac{y^2}{4\sqrt{\lambda}}}\right)^2. \tag{3.9}$$

In the strong coupling limit the two leading terms of $\mathcal{H}^2$ are[5]

$$\mathcal{H}^2\approx\frac{\lambda}{J^2}(p_z z'+p_y y')^2+\sqrt{\lambda}\left(\mathcal{H}_{\rm flat}+\frac{(z^2-y^2)(p_z z'+p_y y')^2}{J^2}\right). \tag{3.10}$$

Here $\mathcal{H}_{\rm flat}=p_z^2+z'^2+p_y^2+y'^2$ is the SO(8) invariant light-cone hamiltonian density for string in flat space. It is clear from eq. (3.10) that there is no well-defined expansion in $1/\sqrt{\lambda}$ neither in the sector of short strings nor in the sector of long strings. Nevertheless, one can see that at strong coupling the energy of short strings scales as $\sqrt[4]{\lambda}$ and the energy of long strings scales as $\sqrt{\lambda}$. First of all we notice that if $m\neq 0$ then the hamiltonian has the following expansion (assuming $m>0$)

$$\mathrm{H}\approx\sqrt{\lambda}m+\frac{1}{2}\int_0^{2\pi}\frac{\mathrm{d}\sigma}{2\pi}\,\frac{\mathcal{H}_{\rm flat}+\frac{1}{J^2}(z^2-y^2)(p_z z'+p_y y')^2}{\frac{1}{J}(p_z z'+p_y y')}\,. \tag{3.11}$$

Due to the non-polynomial structure of the second term the expansion cannot be used in practice to develop perturbation theory in $1/\sqrt{\lambda}$. However, one can see that the contribution of the second term is subleading, and, therefore, the energy of a generic long string scales as $\sqrt{\lambda}m$ in the strong coupling limit.

For $m=0$ even such an expansion as eq. (3.11) becomes impossible. The hamiltonian takes the form

$$\mathrm{H}\approx\int_0^{2\pi}\frac{\mathrm{d}\sigma}{2\pi}\sqrt{\frac{\lambda}{J^2}(p_z z'+p_y y')^2+\sqrt{\lambda}\left(\mathcal{H}_{\rm flat}+\frac{(z^2-y^2)(p_z z'+p_y y')^2}{J^2}\right)}\,. \tag{3.12}$$

This time the first term under the square root in eq. (3.12) is not the leading one. Indeed, if we drop the second term, take the root, and integrate over σ we get zero. We conclude, therefore, that in the strong coupling limit the energy of a generic short string scales as $\sqrt[4]{\lambda}$.

4. The Lax representation

In this section we construct the Lax (zero-curvature) representation for the physical hamiltonian eqs. (2.9), (2.10), proving therefore that it defines a classical integrable system.

[5] We thank Arkady Tseytlin for an important discussion of this point.

The space-time we consider is a coset

$$\mathrm{AdS}_5 \times \mathrm{S}^5 = \mathrm{SO}(4,2)\mathrm{SO}(6)/\mathrm{SO}(4,1) \times \mathrm{SO}(5)$$

and, therefore, the string sigma model must be intimately connected to sigma models on group and coset manifolds. Since fixing the uniform gauge leads to appearance of the gravitational field on the world-sheet it is natural to start with considering the principle sigma model in presence of a non-trivial two-dimensional metric. The field variable of the model is a matrix g and the action reads as

$$\mathrm{S} = \frac{1}{2}\int \mathrm{d}\tau \mathrm{d}\sigma \gamma^{\alpha\beta}\,\mathrm{Tr}\left(\partial_\alpha g g^{-1}\partial_\beta g g^{-1}\right). \tag{4.1}$$

In the case of the flat world-sheet metric integrability of this model is a well-studied subject [40, 41] and it is based on constructing the zero-curvature (Lax) representation for the equations of motion (see also [42]). It is not difficult to generalize this construction to the case of an arbitrary world-sheet metric.

To construct the Lax representation for the principle sigma model with $\gamma^{\alpha\beta}$ arbitrary (but satisfying $\det\gamma = -1$) let us introduce a current A_α (here α is σ or τ):

$$A_\alpha = \partial_\alpha g g^{-1}$$

and its self- and anti-self dual projections

$$A^{\pm}_\alpha = (P^{\pm})_\alpha^{\ \beta} A_\beta\,, \qquad (P^{\pm})_\alpha^{\ \beta} = \delta_\alpha^{\ \beta} \mp \gamma_{\alpha\rho}\epsilon^{\rho\beta}\,. \tag{4.2}$$

Defining the Lax operator which depends on a spectral parameter x as

$$D_\alpha = \partial_\alpha - \frac{A^+_\alpha}{2(1-x)} - \frac{A^-_\alpha}{2(1+x)} \equiv \partial_\alpha - \mathcal{A}_\alpha(x) \tag{4.3}$$

one can verify that the equations of motion for the matrix field g with $\gamma^{\alpha\beta}$ arbitrary but independent of g

$$\partial_\alpha(\gamma^{\alpha\beta}A_\beta) = 0 \tag{4.4}$$

are equivalent to the zero curvature condition

$$[D_\alpha, D_\beta] = 0\,. \tag{4.5}$$

The zero curvature condition, however, does not lead to the Virasoro constraints that follow from varying the world-sheet metric $\gamma^{\alpha\beta}$, and they have to be imposed in addition to (4.5).

It is now easy to generalize this construction to the coset space in hand. Let us introduce the following matrix g

$$g = \begin{pmatrix} g_a & 0 \\ 0 & g_s \end{pmatrix}. \tag{4.6}$$

Here g_a and g_s are the following 4×4 matrices (*cf.* the second paper of [6])

$$g_a = \begin{pmatrix} 0 & \mathcal{Z}_3 & -\mathcal{Z}_2 & \mathcal{Z}_1^* \\ -\mathcal{Z}_3 & 0 & \mathcal{Z}_1 & \mathcal{Z}_2^* \\ \mathcal{Z}_2 & -\mathcal{Z}_1 & 0 & -\mathcal{Z}_3^* \\ -\mathcal{Z}_1^* & -\mathcal{Z}_2^* & \mathcal{Z}_3^* & 0 \end{pmatrix}, \qquad g_s = \begin{pmatrix} 0 & \mathcal{Y}_1 & -\mathcal{Y}_2 & \mathcal{Y}_3^* \\ -\mathcal{Y}_1 & 0 & \mathcal{Y}_3 & \mathcal{Y}_2^* \\ \mathcal{Y}_2 & -\mathcal{Y}_3 & 0 & \mathcal{Y}_1^* \\ -\mathcal{Y}_3^* & -\mathcal{Y}_2^* & -\mathcal{Y}_1^* & 0 \end{pmatrix}. \tag{4.7}$$

To define these matrices we use the complex embedding coordinates $\mathcal{Z}_k$, $k = 1, 2, 3$ for the AdS space and $\mathcal{Y}_k$ for the sphere. Let us discuss the properties of these matrices in more detail.

The matrix g_a is an element of the group $\mathrm{SU}(2, 2)$, *i.e.* it obeys

$$g_a^\dagger E g_a = E\,, \qquad E = \mathrm{diag}(-1, -1, 1, 1)\,, \tag{4.8}$$

provided the following condition is satisfied

$$\mathcal{Z}_1^* \mathcal{Z}_1 + \mathcal{Z}_2^* \mathcal{Z}_2 - \mathcal{Z}_3^* \mathcal{Z}_3 = -1\,. \tag{4.9}$$

In fact g_a describes an embedding of an element of the coset space $\mathrm{SO}(4, 2)/\mathrm{SO}(4, 1)$ into the group $\mathrm{SU}(2, 2)$ which is locally isomorphic to $\mathrm{SO}(4, 2)$. We use this isomorphism to work with 4×4 matrices rather then with 6×6 ones. Quite analogously, g_s is unitary: $g_s g_s^\dagger = 1$ given that the embedding fields satisfy $\mathcal{Y}_k^* \mathcal{Y}_k = 1$. This matrix describes an embedding of an element of the coset $\mathrm{SO}(6)/\mathrm{SO}(5)$ into $\mathrm{SU}(4)$ the latter being isomorphic to $\mathrm{SO}(6)$.

The next step consists in expressing the embedding coordinates $\mathcal{Z}_k$ and $\mathcal{Y}_k$ in terms of physical coordinates and momenta. In particular, $\mathcal{Y}_3$ contains the unphysical field ϕ whose evolution equation is

$$\dot{\phi} = \frac{G_{tt}}{\mathcal{H}} \left(\frac{J}{G_{\phi\phi}} - \frac{\lambda}{J^3} G_{\phi\phi} (p_z z + p_y y)^2 \right). \tag{4.10}$$

Thus, we have a pair of differential equations, (2.6) and (4.10), to determine ϕ via the physical variables. Integrating (2.6) we get

$$\phi(\sigma, \tau) = \phi(0, \tau) - \frac{2\pi}{J} \int_0^\sigma \frac{\mathrm{d}\zeta}{2\pi} (p_z z' + p_y y')\,. \tag{4.11}$$

Here $\phi(0, \tau)$ can be found (up to time-independent constant) by substituting eq. (4.11) into eq. (4.10). Even without solving for $\phi(0, \tau)$ we observe that the field $\phi(\sigma, \tau)$ possesses a non-trivial monodromy

$$\phi(2\pi, \tau) - \phi(0, \tau) = -\frac{2\pi}{J} \mathcal{V}\,. \tag{4.12}$$

As the consequence, the matrix g_s, and, therefore, g have the monodromy which can be written in the form

$$g_s(2\pi) = M g_s(0) M\,, \tag{4.13}$$

where M is a diagonal matrix

$$M = \mathrm{diag}\Big(e^{\frac{i\pi}{J}\mathcal{V}}, e^{-\frac{i\pi}{J}\mathcal{V}}, e^{-\frac{i\pi}{J}\mathcal{V}}, e^{\frac{i\pi}{J}\mathcal{V}} \Big)\,. \tag{4.14}$$

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Using the group element g expressed in terms of physical coordinates and momenta we construct the Lax connection (4.3) which is also block-diagonal

$$\mathcal{A}_\alpha = \begin{pmatrix} \mathcal{A}^a_\alpha & 0 \\ 0 & \mathcal{A}^s_\alpha \end{pmatrix}. \tag{4.15}$$

Now we come to the most important point of our construction. First we notice that the equations of motion for physical fields that follow from the hamiltonian (2.9) obtained by fixing the uniform gauge and solving the Virasoro constraints differ from the dynamical equations (4.4) for the matrix field g only by terms which vanish on the uniform gauge and Virasoro constraints surface. Then it is clear that the zero curvature condition (4.5) for the Lax connection (4.15) again gives the equations (4.4) with the group element g expressed in terms of physical fields, and the world-sheet metric (2.14), (2.18) compatible with the uniform gauge. Therefore, we conclude that the Lax connection (4.15) should lead to the equations of motion that follow from the gauge-fixed hamiltonian (2.9). Indeed, one can check that the dynamical equations (4.4) are identically satisfied provided that we use for the world-sheet metric our solution (2.14), (2.18) and differentiate ϕ according to eqs. (2.6) and (4.10). The calculation is straightforward but rather tedious and, therefore, we refrain from presenting it here. Perhaps, a simplified proof can be found by using the approach of [43]. Thus, we have found the Lax representation for the hamiltonian (2.9), (2.10).

Let us note that the Lax connection we obtained is non-local because it explicitly contains the non-local field ϕ. Moreover, the sphere component $\mathcal{A}^s_\alpha$ of the current $\mathcal{A}_\alpha$ is a quasi-periodic function of σ,

$$\mathcal{A}^s_\alpha(2\pi) = M\mathcal{A}^s_\alpha(0)M^{-1}\,. \tag{4.16}$$

These both pathologies can be cured as follows. Consideration of the structure of $\mathcal{A}_\alpha$ shows that the current can be written in the following way

$$\mathcal{A}^s_\alpha = M(\sigma)\hat{\mathcal{A}}^s_\alpha M(\sigma)^{-1}\,, \tag{4.17}$$

where the σ-dependent matrix M is given by

$$M(\sigma) = \mathrm{diag}\Big(e^{-\frac{i}{2}\phi(\sigma,\tau)}, e^{\frac{i}{2}\phi(\sigma,\tau)}, e^{\frac{i}{2}\phi(\sigma,\tau)}, e^{-\frac{i}{2}\phi(\sigma,\tau)}\Big) \tag{4.18}$$

and $\hat{\mathcal{A}}^s_\alpha$ is local (does not contain $\phi(\sigma,\tau)$). It is worth mentioning that $\hat{\mathcal{A}}^s_\alpha$ is not a connection because $\mathcal{A}^s_\alpha$ and $\hat{\mathcal{A}}^s_\alpha$ are related by a similarity transformation which is not a gauge transformation.

Since the zero-curvature representation (4.5) is invariant under gauge transformations we can gauge the non-local ϕ-dependence away. The local Lax connection arising in this way differs from $\hat{\mathcal{A}}^s_\alpha$ by an additional term coming from the inhomogeneous part of the gauge transformation, and is given by

$$\mathscr{L}^s_\alpha = M^{-1}(\sigma)\mathcal{A}^s_\alpha M(\sigma) - M^{-1}(\sigma)\partial_\alpha M(\sigma) = \hat{\mathcal{A}}^s_\alpha + \frac{i}{2}\partial_\alpha\phi\,\Omega\,, \tag{4.19}$$

where $\Omega = \mathrm{diag}(1,-1,-1,1)$. Here the derivatives of ϕ should be substituted from eqs. (2.6) and (4.10). The new Lax connection is a periodic function of σ since it is a local expression in terms of periodic string coordinates. Let us also note that the original Lax connection (4.3) has poles at $x = \pm 1$ and vanishes at infinity. The gauge transformed connection (4.19) has the same poles but does not vanish at infinity. In particular, the Lax component $\mathscr{L}^s_\sigma$ which will be used in the next section has the following structure

$$\mathscr{L}^s_\sigma(x) = \frac{\mathscr{L}^+}{2(1-x)} + \frac{\mathscr{L}^-}{2(1+x)} - \frac{i}{2J}(p_z z' + p_y y')\Omega \,. \tag{4.20}$$

The (left) Lax representation we consider has also a dual formulation in terms of the right conserved currents $\mathcal{R}^\alpha = -\gamma^{\alpha\beta} g^{-1}\partial_\beta g$. The relation between these two formulations is a gauge transformation by the group element g together with the change $x \to 1/x$ [45]

$$g^{-1} D_\alpha g = \partial_\alpha - \frac{\mathcal{R}^+_\alpha}{2\left(1-\frac{1}{x}\right)} - \frac{\mathcal{R}^-_\alpha}{2\left(1+\frac{1}{x}\right)} \,, \tag{4.21}$$

where the Lax operator D_α is given by (4.3).

5. General properties of monodromy

An important object in the theory of integrable systems is the monodromy matrix $\mathrm{T}(x)$. It is defined as the path-ordered exponential of the Lax component $\mathscr{L}_\sigma(x)$

$$\mathrm{T}(x) = \mathscr{P}\exp\int_0^{2\pi} \mathrm{d}\sigma\; \mathscr{L}_\sigma(x) \,. \tag{5.1}$$

The key property of the monodromy matrix is the time conservation of all its spectral invariants. The trace $\mathrm{Tr}\,\mathrm{T}(x)$, in particular, generates an infinite set of integrals of motion.[6] This stems from the fact that the time evolution of the monodromy is of the Heisenberg type

$$\dot{\mathrm{T}}(x) = [\mathscr{L}_\tau(0,\tau), \mathrm{T}(x)] \,. \tag{5.2}$$

We can also compute the Poisson bracket of $\mathrm{T}(x)$ with the constraint $\mathcal{V}$. Using the definition of $\mathrm{T}(x)$ one first finds

$$\{\mathcal{V}, \mathrm{T}(x)\} = \int_0^{2\pi} \mathrm{d}\sigma' \, \left(\mathscr{P}\exp\int_{\sigma'}^{2\pi} \mathscr{L}_\sigma\right) \{\mathcal{V}, \mathscr{L}_\sigma(\sigma',\tau)\} \left(\mathscr{P}\exp\int_0^{\sigma'} \mathscr{L}_\sigma\right).$$

Recalling eqs. (2.20) and integrating by parts one arrives at

$$\{\mathcal{V}, \mathrm{T}(x)\} = [\mathscr{L}_\sigma(0,\tau), \mathrm{T}(x)] \,. \tag{5.3}$$

[6] One can also define the monodromy matrix by using the quasi-periodic Lax connection $\mathcal{A}_\sigma$: $\mathrm{T}_{\mathrm{nl}}(x) = \mathscr{P}\exp\int_0^{2\pi}\mathrm{d}\sigma\; \mathcal{A}_\sigma(x)$. However, the spectral invariants of T_{nl} and T will coincide only on solutions of constraint (2.7). In general the spectral invariants of $\mathrm{T}_{\mathrm{nl}}(x)$ are not conserved.

Thus, trace of the monodromy as well as all its spectral invariants also Poisson commute with the remaining constraint. The Jacobi identity $\{\mathrm{H}, \{\mathcal{V}, \mathrm{T}\}\}$ + perm. is satisfied by virtue of the Lax representation for $\mathscr{L}$.

To study the analytic properties of the monodromy it is useful to denote the eigenvalues of T as $\exp(ip_k(x))$, where in our case $k = 1, \ldots, 8$. The function $p_k(x)$ is known as the quasi-momentum (the Floquet function) and it plays an important role in the (quantum) inverse scattering method [44].

As is well known in the theory of integrable PDEs, the *local* integrals of motion are obtained by expanding the eigenvalues of $\mathrm{T}(x)$ around the poles of the Lax connection, which in our case are at $x = \pm 1$. It is, therefore, interesting to look at the first non-trivial integral arising in the expansion around $x = \pm 1$.

Around $x \to 1$ one can use a regular gauge transformation to bring $\mathscr{L}^+$ to the diagonal form (up to permutations of eigenvalues). We find the following result

$$\mathscr{L}^+ \to \frac{i}{2\sqrt{\lambda}} \mathrm{diag}\Big(\underbrace{\kappa_+, -\kappa_+, \kappa_+, -\kappa_+}_{\mathrm{AdS}} ; \underbrace{\kappa_+, -\kappa_+, \kappa_+, -\kappa_+}_{\mathrm{Sphere}} \Big) . \tag{5.4}$$

The fact that all the eigenvalues appear to be proportional to one and the same value κ_+ is a consequence of a peculiar form of the matrices g_a and g_s (and the associated currents $\mathscr{L}_\alpha$) — they have a special property of being skew-symmetric. After some computation we find

$$\kappa_+^2 = \frac{J^2}{G_{\phi\phi}} + \frac{\lambda}{J^2} G_{\phi\phi} (p_y y' + p_z z')^2 - 2\sqrt{\lambda}\ (p_z z') + \left(1 + \frac{y^2}{4}\right)^2 p_y^2 + \frac{\lambda y'^2}{(1 + \frac{y^2}{4})^2} .$$

This expression can be also rewritten in a more compact form

$$\kappa_+^2 = \frac{\mathcal{H}^2}{G_{tt}} - \left[\left(1 - \frac{z^2}{4}\right) p_{z_k} + \frac{\sqrt{\lambda}\ z'_k}{(1 - \frac{z^2}{4})} \right]^2 . \tag{5.5}$$

Given this result one can directly verify that the integral

$$\int_0^{2\pi} \frac{\mathrm{d}\sigma}{2\pi} \kappa_+ \tag{5.6}$$

is conserved. Interestingly enough it does not coincide with the hamiltonian H. However, if we reduce the classical string theory to the one on $\mathbb{R} \times \mathrm{S}^5$, which amounts to putting $z_k = 0 = p_{z_k}$, this integral becomes the string hamiltonian.

In complete analogy to the previous consideration we determine the asymptotic behavior of the monodromy around $x = -1$. We find

$$\mathscr{L}^- \to \frac{i}{2\sqrt{\lambda}} \mathrm{diag}\Big(\underbrace{\kappa_-, -\kappa_-, \kappa_-, -\kappa_-}_{\mathrm{AdS}} ; \underbrace{\kappa_-, -\kappa_-, \kappa_-, -\kappa_-}_{\mathrm{Sphere}} \Big) , \tag{5.7}$$

where κ_- is

$$\kappa_-^2 = \frac{\mathcal{H}^2}{G_{tt}} - \left[\left(1-\frac{z^2}{4}\right)p_{z_k} - \frac{\sqrt{\lambda}\, z_k'}{(1-\frac{z^2}{4})}\right]^2 . \tag{5.8}$$

Thus, we observe that two infinite series of local integrals of motion obtained by expanding the Lax connection around two different poles are different, they merge under the reduction of string motion to $\mathbb{R}\times S^5$. Since trace of the monodromy matrix is gauge invariant our results are not specific to the uniform gauge we use but also hold, e.g., in the conformal gauge. Therefore, in the general case of $AdS_5 \times S^5$ the asymptotic behavior of quasi-momentum around $x = \pm 1$ is not related to the string energy H or the other global charges in an obvious way, quite opposite to what happens in cases of string theory on $\mathbb{R}\times S^3$ and $\mathbb{R}\times S^5$ [7].

Finally, it is interesting to consider the reduction to $AdS_5 \times S^1$, which corresponds to taking all $y_k = 0 = p_{y_k}$. Remarkably, in this case the $\kappa_\pm^2$ become the perfect squares and we therefore obtain

$$\kappa_\pm = J \mp \frac{\sqrt{\lambda}}{J} p_z z' . \tag{5.9}$$

Thus, around $x \to \pm 1$ the quasi-moment behaves as (up to the sign ambiguity)

$$p(x) = \frac{1}{x\mp 1}\int_0^{2\pi} \frac{d\sigma}{2\sqrt{\lambda}}\left(J \mp \frac{\sqrt{\lambda}}{J}p_z z'\right) + \cdots = \frac{\pi}{x \mp 1}\left(\frac{J}{\sqrt{\lambda}} \mp m\right) + \cdots , \tag{5.10}$$

where we made use of the constraint (2.7). This asymptotic expansion perfectly agrees with the one obtained for the string sigma model on $AdS_3 \times S^1$ by Kazakov and Zarembo [7].

To complete our discussion of the asymptotic properties of the quasi-momentum, we also exhibit, in the spirit of [7], the asymptotic behavior of $p_k(x)$ around $x \to 0$ and $x \to \infty$. To this end we assume that the classical solutions we consider carry only the Cartan (abelian) charges of the unbroken symmetry group $SO(4)\times SO(4)$: two AdS charges S_1 and S_2, and another two charges J_1 and J_2, which together with J are the angular momentum components of string rotating in S^5. The explicit form of these charges in terms of physical variables is given in appendix A. We also assume the fulfillment of the constraint (2.7).

To find the asymptotics of the quasi-momenta around $x \to \infty$ we expand the monodromy matrix around this (regular) point

$$T(x) = \mathbb{I} + \frac{1}{x}\int d\sigma\left(\gamma^{\tau\sigma}A_\sigma + \gamma^{\tau\tau}A_\tau\right) + \cdots . \tag{5.11}$$

Here we used eqs. (4.2) and the fact that on the solutions of the Virasoro constraint the spectral invariants of T and T_{nl} coincide. The integral in eq. (5.11) is the (matrix) Noether charge corresponding to the global symmetry transformations $g \to hg$. Since solutions we consider carry only the Cartan charges of the global symmetry group the matrix of the

Noether charges is diagonal. Using the explicit form of the diagonal matrix elements of $\gamma^{\tau\alpha}A_\alpha$ we find that the individual quasi-momenta exhibit the following asymptotics (up to shifts by integer multiples of 2π)

$$\begin{aligned}
&\qquad\qquad \text{AdS} && \qquad\qquad \text{Sphere} \\
p_1(x) &= \frac{2\pi}{x\sqrt{\lambda}}(-\mathrm{H}-S_1+S_2)\,, & p_5(x) &= \frac{2\pi}{x\sqrt{\lambda}}(J-J_1-J_2)\,, \\
p_2(x) &= \frac{2\pi}{x\sqrt{\lambda}}(-\mathrm{H}+S_1-S_2)\,, & p_6(x) &= \frac{2\pi}{x\sqrt{\lambda}}(-J-J_1+J_2)\,, \\
p_3(x) &= \frac{2\pi}{x\sqrt{\lambda}}(\mathrm{H}+S_1+S_2)\,, & p_7(x) &= \frac{2\pi}{x\sqrt{\lambda}}(-J+J_1-J_2)\,, \\
p_4(x) &= \frac{2\pi}{x\sqrt{\lambda}}(\mathrm{H}-S_1-S_2)\,, & p_8(x) &= \frac{2\pi}{x\sqrt{\lambda}}(J+J_1+J_2)\,.
\end{aligned}$$

Analogous consideration can be performed to find asymptotics around $x \to 0$. To guarantee that expansion of the monodromy around $x \to 0$ starts from identity one can use the formula (4.21) which relates the asymptotics of $\mathscr{L}$ around zero with the asymptotics of $\frac{x}{2(x-1)}\mathcal{R}^+_\alpha + \frac{x}{2(x+1)}\mathcal{R}^-_\alpha$ around infinity. The Noether charge arising around $x \to 0$ is related to the global symmetry transformations $g \to hg$. Requiring absence of non-diagonal components for this charge allows one to find

$$\begin{aligned}
&\qquad\qquad \text{AdS} && \qquad\qquad \text{Sphere} \\
p_1(x) &= x\frac{2\pi}{\sqrt{\lambda}}(\mathrm{H}+S_1-S_2)\,, & p_5(x) &= x\frac{2\pi}{\sqrt{\lambda}}(-J+J_1+J_2)\,, \\
p_2(x) &= x\frac{2\pi}{\sqrt{\lambda}}(\mathrm{H}-S_1+S_2)\,, & p_6(x) &= x\frac{2\pi}{\sqrt{\lambda}}(J+J_1-J_2)\,, \\
p_3(x) &= x\frac{2\pi}{\sqrt{\lambda}}(-\mathrm{H}-S_1-S_2)\,, & p_7(x) &= x\frac{2\pi}{\sqrt{\lambda}}(J-J_1+J_2)\,, \\
p_4(x) &= x\frac{2\pi}{\sqrt{\lambda}}(-\mathrm{H}+S_1+S_2)\,, & p_8(x) &= x\frac{2\pi}{\sqrt{\lambda}}(-J-J_1-J_2)\,.
\end{aligned}$$

Actually, the relation between the asymptotics of quasi-momenta around $x \to 0$ and $x \to \infty$ (and also around $x \to \pm 1$) is determined only up to the permutation ambiguity. One of the possible choices of $p_k(x)$ can be done, for instance, by requiring that in the absence of winding (i.e. when $m = 0$ in (2.7)) the following relation is satisfied $p_k(1/x) = -p_k(x)$. The sign "-" here is due to relation (4.21).

Finally, for the trace of the monodromy matrix around $x \to 0$ we get

$$\operatorname{Tr}\mathrm{T}(x) = 8 - x^2\frac{8\pi^2}{\lambda}\Big[\mathrm{H}^2 + S_1^2 + S_2^2 + J^2 + J_1^2 + J_2^2\Big] + \cdots\,, \tag{5.12}$$

while around $x \to \infty$ we obtain

$$\operatorname{Tr}\mathrm{T}(x) = 8 - \frac{8\pi^2}{x^2\lambda}\Big[\mathrm{H}^2 + S_1^2 + S_2^2 + J^2 + J_1^2 + J_2^2\Big] + \cdots\,, \tag{5.13}$$

In principle, it is now straightforward to generalize the results obtained in [7], and construct the classical Bethe equations for the string theory on $\mathrm{AdS}_5 \times \mathrm{S}^1$. Due to the complicated asymptotic behavior of quasi-momentum around $x \to \pm 1$, eqs. (5.5) and (5.8), the challenge, however, is to derive the equations for the string theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$.

6. Conclusions

In this paper we developed the hamiltonian formalism for classical strings on $\mathrm{AdS}_5 \times \mathrm{S}^5$. The hamiltonian is obtained in the uniform gauge and depends on two parameters: the S^5 angular momentum component J and the string tension λ. In the large J expansion with the effective BMN coupling λ' kept fixed the hamiltonian reproduces the plane-wave hamiltonian and higher corrections previously found in [25, 39]. We then exhibited kinematical integrability of the hamiltonian (for J and λ finite) by explicitly constructing the corresponding Lax representation. In this respect we note that emergence of an integrable structure is rather intricate because the hamiltonian turns out to be of a non-polynomial (Nambu) type. We further verified that the asymptotic properties of the quasi-momentum (the generating function of the integrals of motion) perfectly agree with the ones obtained earlier for some specific cases [7].

Let us now formulate several open problems naturally arising in our approach. As we have seen, for the general $\mathrm{AdS}_5 \times \mathrm{S}^5$ model the asymptotics of the quasi-momentum around $x \to \pm 1$ is not related to the global conserved charges in a simple way. This appears to be an obstacle in formulating the classical string Bethe equations in full generality. To get more insight into this problem it is desirable to analyze the higher local conserved charges arising in the expansion around $x \to \pm 1$. Alternatively, the local integrals of motion for the string sigma model can be found by means of the Bäcklund transform [16]. It is, therefore, interesting to construct the Bäcklund equations for the sigma model coupled to 2d gravity and analyze the corresponding conservation laws.

The knowledge of the continuous string Bethe equations can be further used to guess the fundamental Bethe equations which would describe the quantum string, at least in some asymptotic expansions [28]. Another way to approach the quantization problem is to find first the separated variables for the classical string hamiltonian (2.9). To this end one should investigate the Poisson structure of the Lax connection $\mathscr{L}$ and establish a relation to the (dynamical) r-matrix approach. We expect, however, that the Poisson structure will not be ultra-local, i.e. it will contain the $\delta'(\sigma - \sigma')$ term, as it appears already for the sigma model in the conformal gauge [45].

To maintain the conformal invariance at the quantum level one needs to include the fermions. We believe that fermionic degrees of freedom can be naturally incorporated in the hamiltonian approach without spoiling the kinematical integrability. In particular, the zero-curvature representation for the Green-Schwarz superstring found in [33] could be of use here. The knowledge of the classical separated variables might help to approach the formidable problem of finding the separated variables in the quantum case, see [46, 47] for interesting examples.

The uniform gauge is not the only gauge one can use to fix the gauge freedom of the string theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$. In particular, choosing the uniform gauge implies non-zero J and, therefore, leads to missing a sector of string states with $J = 0$. Another interesting gauge condition is the AdS_5 light-cone gauge proposed in [37]. An important advantage of this gauge is that fermions as well as spinless string states can be readily taken into account. It would be very interesting to use our method to derive a Lax pair for the AdS_5 light-cone hamiltonian obtained in [37].

Finally, it is of interest to clarify the relation between the exact Lax pair we constructed here and the perturbative $1/J$ Lax pair recently obtained in [32].

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A. Equations of motion and charges

Equations of motion for physical variables generated by the hamiltonian (2.9) are

$$
\begin{aligned}
\dot{z}_k &= \frac{G_{tt}}{\mathcal{H}}\left[\left(1-\frac{z^2}{4}\right)^2 p_{z_k} + \frac{\lambda}{J^2}G_{\phi\phi}(p_z z' + p_y y')z'_k\right],\\
\dot{y}_k &= \frac{G_{tt}}{\mathcal{H}}\left[\left(1+\frac{y^2}{4}\right)^2 p_{y_k} + \frac{\lambda}{J^2}G_{\phi\phi}(p_z z' + p_y y')y'_k\right],\\
\dot{p}_{z_k} &= \frac{z_k}{\mathcal{H}(1-\frac{z^2}{4})}\left[-\frac{\mathcal{H}^2}{(1+\frac{z^2}{4})} + \frac{1}{2}\left(1-\frac{z^2}{4}\right)^2 p_z^2 - \frac{\lambda z'^2}{2(1-\frac{z^2}{4})^2}\right] +\\
&\quad +\partial_\sigma\left[\frac{G_{tt}}{\mathcal{H}}\left(\frac{\lambda}{J^2}G_{\phi\phi}(p_z z' + p_y y')p_{z_k} + \frac{\lambda}{(1-\frac{z^2}{4})^2}z'_k\right)\right],\\
\dot{p}_{y_k} &= -y_k\frac{G_{tt}}{\mathcal{H}(1+\frac{y^2}{4})(1-\frac{y^2}{4})}\times\\
&\quad\times\left[\frac{J^2}{G_{\phi\phi}} - \frac{\lambda}{J^2}G_{\phi\phi}(p_z z' + p_y y')^2 + \frac{1}{2}\left(1-\frac{y^2}{4}\right)\left(\left(1+\frac{y^2}{4}\right)^2 p_y^2 - \frac{\lambda y'^2}{(1+\frac{y^2}{4})^2}\right)\right] +\\
&\quad +\partial_\sigma\left[\frac{G_{tt}}{\mathcal{H}}\left(\frac{\lambda}{J^2}G_{\phi\phi}(p_z z' + p_y y')p_{y_k} + \frac{\lambda}{(1+\frac{y^2}{4})^2}y'_k\right)\right].
\end{aligned}
\tag{A.1}
$$

In particular, the first two equations are used to eliminate $\dot{z}_k$ and $\dot{y}_k$ from the current $\mathcal{A}_\alpha$ in favor of the corresponding canonical momenta p_{z_k} and p_{y_k}.

The hamiltonian (2.9) has $SO(4)\times SO(4)$ symmetry. It is generated by the following charges: the Cartan generators S_1 and S_2 for the SO(4) symmetry rotating the AdS coordinates are

$$
S_1 = \int_0^{2\pi}\frac{d\sigma}{2\pi}z_{[1}p_{z_{2]}}, \qquad S_2 = \int_0^{2\pi}\frac{d\sigma}{2\pi}z_{[3}p_{z_{4]}}, \tag{A.2}
$$

while the Cartan generators J_1 and J_2 for the SO(4) group acting on the coordinates of the sphere have the form

$$J_1 = \int_0^{2\pi} \frac{\mathrm{d}\sigma}{2\pi} y_{[1} p_{y_2]}\,, \qquad J_2 = \int_0^{2\pi} \frac{\mathrm{d}\sigma}{2\pi} y_{[3} p_{y_4]}\,. \tag{A.3}$$

In these formulae $y_{[1}p_{y_2]} = y_1 p_{y_2} - y_2 p_{y_1}$, etc.

B. Strings on $\mathbb{R} \times \mathrm{S}^3$

As we have seen the string hamiltonian and its Lax representation are rather complicated. To approach a difficult problem of finding the separated variables for the hamiltonian (and their subsequent quantization) one could start from a smaller subsector, e.g., from string theory on $\mathbb{R} \times \mathrm{S}^3$. This model is not conformal at the quantum level but hopefully it remains integrable. Unraveling its integrable structure might provide further insight on the general problem. In this appendix we collect the relevant formulae to analyze strings on $\mathbb{R} \times \mathrm{S}^3$ in the hamiltonian setting.

Reduction to $\mathbb{R} \times \mathrm{S}^3$ consists in taking $z_k = 0 = p_{z_k}$ for all $k = 1, \ldots 4$ and $y_3 = y_4 = p_{y_3} = p_{y_4} = 0$. Thus, the physical variables are two coordinates, y_1 and y_2, and their conjugate momenta p_1 and p_2. The square of the hamiltonian density becomes

$$\mathcal{H}^2 = \frac{J^2}{G_{\phi\phi}} + \frac{\lambda}{J^2} G_{\phi\phi} (p_y y')^2 + \left(1 + \frac{y^2}{4}\right)^2 p_y^2 + \frac{\lambda y'^2}{(1+\frac{y^2}{4})^2}\,.$$

The corresponding Lax connection can be written in terms of 2×2 matrices. For instance, the σ-component reads as

$$\mathscr{L}_\sigma = \frac{\mathscr{L}^+}{2(1-x)} + \frac{\mathscr{L}^-}{2(1+x)} - \frac{i}{2J}(p_y y')\sigma_3\,, \tag{B.1}$$

where σ_3 is the Pauli matrix and

$$\mathscr{L}^\pm = \begin{pmatrix} \mathscr{L}_{11}^\pm & \mathscr{L}_{12}^\pm \\ -\mathscr{L}_{12}^{\pm *} & -\mathscr{L}_{11}^\pm \end{pmatrix}.$$

For $\mathscr{L}^+$ we have

$$\begin{aligned} \mathscr{L}_{11}^+ &= -\frac{i}{\sqrt{\lambda}} J + \frac{i}{J} G_{\phi\phi}(py') + \frac{i}{\sqrt{\lambda}} y_{[1}p_{2]} + \frac{i}{(1+\frac{y^2}{4})^2} y_{[1}y'_{2]}\,, \\ \mathscr{L}_{12}^+ &= i\frac{y_1 + i y_2}{1 - \frac{y^2}{4}} \left[-\frac{J}{\sqrt{\lambda}} + \frac{G_{\phi\phi}}{J}(py') \right] - \\ &\quad - \left[\frac{1}{\sqrt{\lambda}}(p_1 + i p_2) + \frac{y'_1 + i y'_2}{(1+\frac{y^2}{4})^2} \right] - \frac{(y_1 + i y_2)^2}{4} \left[\frac{1}{\sqrt{\lambda}}(p_1 - i p_2) + \frac{y'_1 - i y'_2}{(1+\frac{y^2}{4})^2} \right] \end{aligned} \tag{B.2}$$

and for $\mathscr{L}^-$

$$\mathscr{L}_{11}^- = \frac{i}{\sqrt{\lambda}} J + \frac{i}{J} G_{\phi\phi}(py') - \frac{i}{\sqrt{\lambda}} y_{[1}p_{2]} + \frac{i}{(1+\frac{y^2}{4})^2} y_{[1}y'_{2]}\,,$$

$$\mathscr{L}_{12}^{-} = i\frac{y_1+iy_2}{1-\frac{y^2}{4}}\left[\frac{J}{\sqrt{\lambda}}+\frac{G_{\phi\phi}}{J}(py')\right]+$$
$$+\left[\frac{1}{\sqrt{\lambda}}(p_1+ip_2)-\frac{y_1'+iy_2'}{(1+\frac{y^2}{4})^2}\right]+\frac{(y_1+iy_2)^2}{4}\left[\frac{1}{\sqrt{\lambda}}(p_1-ip_2)-\frac{y_1'-iy_2'}{(1+\frac{y^2}{4})^2}\right] \quad \text{(B.3)}$$

In these formulae $y_{[1}p_{y_2]} = y_1p_{y_2} - y_2p_{y_1}$, etc.

Expansion around the plane-wave limit is constructed by rescaling the coordinates and momenta according to eqs. (3.1). It is not difficult to compute the monodromy perturbatively in $1/\sqrt{J}$. For instance, for the matrix elements of T

$$\mathrm{T} = \begin{pmatrix} \mathrm{T}_{11} & \mathrm{T}_{12} \\ \mathrm{T}_{21} & \mathrm{T}_{22} \end{pmatrix}$$

up to the order $1/J$ we find

$$\mathrm{T}_{11} = e^{-i\pi\omega} - \frac{e^{-i\pi\omega}}{J}A\,, \qquad \mathrm{T}_{22} = e^{i\pi\omega} - \frac{e^{i\pi\omega}}{J}A^*$$

and

$$\mathrm{T}_{12} = \frac{e^{-i\pi\omega}}{\sqrt{J}}\int_0^{2\pi}\mathrm{d}\sigma b(\sigma)e^{i\omega\sigma}\,, \qquad \mathrm{T}_{21} = -\frac{e^{i\pi\omega}}{\sqrt{J}}\int_0^{2\pi}\mathrm{d}\sigma b^*(\sigma)e^{-i\pi\omega\sigma}\,.$$

Here we use the notation

$$A = \int_0^{2\pi}\mathrm{d}\sigma\int_0^{\sigma}\mathrm{d}\sigma'\; b(\sigma)b^*(\sigma')e^{i\omega(\sigma-\sigma')} - \int_0^{2\pi}\mathrm{d}\sigma c(\sigma)\,,$$

where the functions $b(\sigma)$ and $c(\sigma)$ are

$$\begin{aligned} b(\sigma) &= \frac{1}{\sqrt{\lambda'}}\left(-\frac{x}{1-x^2}(i(y_1+iy_2)+p_1+ip_2)-\frac{\sqrt{\lambda'}}{1-x^2}(y_1'+iy_2')\right)\,,\\ c(\sigma) &= \frac{i}{\sqrt{\lambda'}}\frac{x}{1-x^2}y_{[1}p_{2]}+\frac{i}{1-x^2}y_{[1}y_{2]}'+\frac{i}{2}\frac{1+x^2}{1-x^2}(py')\,. \end{aligned}$$

and we have used the concise notation

$$\omega = \frac{2}{\sqrt{\lambda'}}\frac{x}{1-x^2}\,. \tag{B.4}$$

In fact, one can consider ω as the new spectral parameter, the map from the x-plane to the ω-plane is two-fold, as

$$x = \frac{-1\pm\sqrt{1+\lambda'\omega^2}}{\sqrt{\lambda'}\omega}\,. \tag{B.5}$$

As was discussed in section 5 the local integrals of motion are obtained by expanding the trace of the monodromy matrix around the singularities of the Lax connection which are at $x=\pm1$. However, analyzing the structure of the monodromy matrix computed perturbatively in $1/J$ one can recognize that the limit $x\to\pm1$ is ill-defined. This clearly shows that two expansions, $x \to \pm 1$ and $J \to \infty$, are not permutable. The model, of course, remains to be integrable, the local charges are reorganized in a different expansion.

From the results above it is easy to see the appearance of the plane-wave physics. The conventional way to solve the periodic integrable model is to use separation of variables [46]. If the Poisson bracket $\{\mathrm{T}_{12}(x), \mathrm{T}_{12}(x')\}$ vanishes then the matrix element $\mathrm{T}_{12}(x)$ can be considered as a new coordinate. Introduce a variable

$$t(\omega) = \int_0^{2\pi} \mathrm{d}\sigma b(\sigma) e^{i\omega\sigma} .$$

By using the equations of motion generated by the plane-wave hamiltonian

$$\dot{y} = p , \qquad \dot{p} = -y + \lambda' y'' ,$$

one can easily check that indeed $\{t(\omega), t(\omega')\} = 0$. Further one finds

$$\{t(\omega), t^*(\omega')\} = \left(- \omega\omega' + \frac{\omega^2}{1-x'^2} + \frac{\omega'^2}{1-x^2} \right) \frac{e^{2\pi i(\omega-\omega')} - 1}{\omega - \omega'} .$$

For ω arbitrary this bracket is not canonical. The canonical bracket arises when ω is an integer. In this case we have

$$\{t(\omega), t^*(\omega')\} = -2\pi i \omega^2 \sqrt{1+\omega^2\lambda'} \delta(\omega-\omega') ,$$

i.e. the canonical (separated) variables are

$$a(\omega) = \frac{t(\omega)}{i\omega\sqrt[4]{1+\omega^2\lambda'}}$$

with the bracket

$$\{a^*(\omega'), a(\omega)\} = -2\pi i \delta(\omega - \omega') .$$

It is also interesting to look at the time evolution of $t(\omega) \equiv t(x)$. Expanding (5.2) in inverse powers of $\sqrt{J}$ we find at leading order

$$\dot{t}(x) = -i\frac{1+x^2}{1-x^2} t(x) - \frac{\sqrt{\lambda'}}{x} b\left(\frac{1}{x}, 0\right) \left(e^{2\pi i\omega} - 1\right) .$$

Here we also exhibit the dependence of the function $b(\sigma)$ on the spectral parameter x. Thus, in the periodic case the dynamics of the coefficient $t(x)$ is not simple and depends on the boundary value of fields at some point. In fact, this is a major obstacle in application of the inverse scattering method to the periodic case. Roughly speaking, for rapidly decreasing fields on a line the role of zero point is played by infinity where the fields vanish and, therefore, the dynamics of the transition coefficients simplifies. In the distinguished case of ω integer, however, the unwanted term containing $b(0)$ disappears and the dynamics of $t(x)$ becomes trivial. In this case the evolution equation can be immediately integrated and we find the BMN type formula

$$t(\omega, \tau) = e^{-i\tau\sqrt{1+\lambda'\omega^2}} t(\omega, 0) , \qquad \omega \in \mathbb{Z} . \tag{B.6}$$

Finally for ω integer we find

$$\operatorname{Tr} \mathrm{T} = (e^{i\pi\omega} + e^{-i\pi\omega})\left(1 - \frac{|t(\omega)|^2}{2J} + \cdots\right). \tag{B.7}$$

One can easily see that the conserved quantity $|t(\omega)|^2$ is nothing else but the density of the plane-wave hamiltonian written in terms of separated variables. The corresponding quasi-momentum $p(\omega) = \arccos(\frac{1}{2}\mathrm{T})$ has an expansion in powers of $1/\sqrt{J}$. Note that for ω non-integer one would get the following $1/J$ expansion

$$p(\omega) = \pi\omega + \frac{e^{-i\pi\omega}A + e^{i\pi\omega}A^*}{2J|\sin(\pi\omega)|} + \cdots . \tag{B.8}$$

This expression does not have the limit $\omega \to$ integer. Therefore, at integer values of ω the expansion of the quasi-moment changes drastically. Instead of $1/J$ expansion at regular (non-integer) values we have $1/\sqrt{J}$ expansion for ω integer-valued.

The perturbative treatment can be pushed to higher orders in $1/\sqrt{J}$. It is however hardly possible that $\mathrm{T}_{12}(x)$ would provide the separated variables: For J finite dynamics of $\mathrm{T}_{12}(x)$ is complicated and depends on value of the fields at zero point.

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